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STABILITY OF BRANCHED PULL-BACK PROJECTIVE FOLIATIONS

COSTA E SILVA, W.

Abstract. We prove that, if $n \geq 3$, a singular foliation $\mathcal{F}$ on $\mathbb{P}^n$ which can be written as pull-back, where $\mathcal{G}$ is a foliation in $\mathbb{P}^2$ of degree $d \geq 2$ with one or three invariant lines in general position and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $deg\,(f) = \nu \geq 2$, is an appropriated rational map, is stable under holomorphic deformations. As a consequence we conclude that the closure of the sets $\{\mathcal{F} = f^*(\mathcal{G})\}$ are new irreducible components of the space of holomorphic foliations of certain degrees.

Contents

1. Introduction

Let $\mathcal{F}$ be a holomorphic singular foliation on $\mathbb{P}^n$ of codimension 1, $\Pi_n : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n$ be the natural projection and $\mathcal{F}^* = \Pi_n^*(\mathcal{F})$. It is known that $\mathcal{F}^*$ can be defined by an integrable $1-$form $\Omega = \sum_{j=0}^{n} A_j dz_j$ where the $A_j's$ are homogeneous polynomials of the same degree $k+1$ satisfying the Euler condition:

$$\sum_{j=0}^{n} z_j A_j \equiv 0. \tag{1.1}$$

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The singular set $S(\mathcal{F})$ is given by $S(\mathcal{F}) = \{A_0 = ... = A_n = 0\}$ and is such that $\mathrm{codim}(S(\mathcal{F})) \geq 2$. The integrability condition is given by

$$\Omega \wedge d\Omega = 0. \tag{1.2}$$

The form Ω will be called a homogeneous expression of $\mathcal{F}$. The degree of $\mathcal{F}$ is, by definition, the number of tangencies (counted with multiplicities) of a generic linearly embedded $\mathbb{P}^1$ with $\mathcal{F}$. If we denote it by $deg(\mathcal{F})$ then $deg(\mathcal{F}) = k$. The set of homogeneous 1-forms which satisfy (1.1) and (1.2) will be denoted by $\tilde{\Omega}^1(n, k+1)$. We denote the space of foliations of a fixed degree k in $\mathbb{P}^n$ by $\mathbb{F}\mathrm{ol}(k,n)$. Due to the integrability condition and the fact that $S(\mathcal{F})$ has codimension ≥ 2, we see that $\mathbb{F}\mathrm{ol}(k,n)$ can be identified with a Zariski's open set in the variety obtained by projectivizing the space of forms Ω which satisfy (1.1) and (1.2), i.e $\mathbb{P}\tilde{\Omega}^1(n, k+1)$. It is in fact an intersection of quadrics. To obtain a satisfactory description of $\mathbb{F}\mathrm{ol}(k;n)$ (for example, to talk about deformations) it would be reasonable to know the decomposition of $\mathbb{F}\mathrm{ol}(k;n)$ in irreducible components. This leads us to the following:

Problem: ***Describe and classify the irreducible components of*** $\mathbb{F}\mathrm{ol}(k;n)$ $k \geq 3$ ***on*** $\mathbb{P}^n$***,*** $n \geq 3$.

One can exhibit some kind of list of components in every degree, but this list is incomplete. In the paper [C.LN1], the authors proved that the space of holomorphic codimension one foliations of degree 2 on $\mathbb{P}^n, n \geq 3$, has six irreducible components, which can be described by geometric and dynamic properties of a generic element. We refer the curious reader to [C.LN1] and [LN0] for a detailed description of them. There are known families of irreducible components in which the typical element is a pull-back of a foliation on $\mathbb{P}^2$ by a rational map. Given a generic rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ of degree $\nu \geq 1$, it can be written in homogeneous coordinates as $f = (F_0, F_1, F_2)$ where F_0, F_1 and F_2 are homogeneous polynomials of degree ν. Now consider a foliation $\mathcal{G}$ on $\mathbb{P}^2$ of degree $d \geq 2$. We can associate to the pair $(f, \mathcal{G})$ the pull-back foliation $\mathcal{F} = f^*\mathcal{G}$. The degree of the foliation $\mathcal{F}$ is $\nu(d+2)-2$ as proved in [C.LN.E]. Denote by $PB(d, \nu; n)$ the closure in $\mathbb{F}\mathrm{ol}(\nu(d+2)-2, n)$, $n \geq 3$ of the set of foliations $\mathcal{F}$ of the form $f^*\mathcal{G}$. Since $(f, \mathcal{G}) \to f^*\mathcal{G}$ is an algebraic parametrization of $PB(d, \nu; n)$ it follows that $PB(d, \nu; n)$ is an unirational irreducible algebraic subset of $\mathbb{F}\mathrm{ol}(\nu(d+2)-2, n)$, $n \geq 3$. We have the following result:

Theorem 1.1. *$PB(d, \nu; n)$ is a unirational irreducible component of $\mathbb{F}\mathrm{ol}(\nu(d+2)-2, n)$; $n \geq 3$, $\nu \geq 1$ and $d \geq 2$.*

The case $\nu = 1$, of linear pull-backs, was proven in [Ca.LN], whereas the case $\nu > 1$, of nonlinear pull-backs, was proved in [C.LN.E]. The search for new components of pull-back type was started in the Ph.D thesis of the author [CS]. There we began to consider branched rational maps and foliations with algebraic invariant sets of positive dimensions.

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$ which can be written as $\mathcal{F} = f^*(\mathcal{G})$, where $\mathcal{G}$ is a foliation in $\mathbb{P}^2$ of degree $d \geq 2$ with three invariant lines in general position, say $(XYZ) = 0$, and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $deg(f) = \nu \geq 2$, $f = \left(F_0^{\alpha} : F_1^{\beta} : F_2^{\gamma}\right)$. Denote by $PB(k, \nu, \alpha, \beta, \gamma)$ the closure in $\mathbb{F}\mathrm{ol}(k, n)$, $n \geq 3$ of the set of foliations $\mathcal{F}$ of the form $f^*\mathcal{G}$. The degree of the foliation $\mathcal{F}$ is $k = \nu\left[(d-1) + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] - 2$, as proved in [CS]. Since $(f, \mathcal{G}) \to f^*\mathcal{G}$ is an algebraic parametrization of

$PB(k,\nu,\alpha,\beta,\gamma)$ it follows that $PB(k,\nu,\alpha,\beta,\gamma)$ is an unirational irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\,(k,n)$, $n \geq 3$. In [CS] we proved the following result:

Theorem 1.2. *$PB(k,\nu,\alpha,\beta,\gamma)$ is a unirational irreducible component of $\mathbb{F}\mathrm{ol}\,(k,n)$ for all $n \geq 3$, $deg(F_0).\alpha = deg(F_1).\beta = deg(F_2).\gamma = \nu \geq 2$, $(\alpha,\beta,\gamma) \in \mathbb{N}^3$ such that $1 < \alpha < \beta < \gamma$ and $d \geq 2$.*

In this paper we continue looking for new components of branched pull back-type. In this direction will extend the previous result to case where $\alpha = \beta \geq 1$. We observe that in the case $\alpha = \beta > 1$ we continue dealing with foliations in $\mathbb{P}^2$ with three invariant lines in general position. On the other hand, in the situation $\alpha = \beta = 1$ we need to consider another set of foliations in $\mathbb{P}^2$. That is, we need foliations with one invariant line. Let us describe this last case: Let $\mathcal{G}$ be a foliation on $\mathbb{P}^2$ with one invariant straight line, say ℓ. Consider coordinates $(X,Y,Z) \in \mathbb{C}^3$ such that $\ell = \Pi_2(Z=0)$, where $\Pi_2 : \mathbb{C}^3 \backslash \{0\} \to \mathbb{P}^2$ is the natural projection. The foliation $\mathcal{G}$ can be represented in these coordinates by a polynomial 1-form of the type $\Omega = ZA\,(X,Y,Z)\,dX + ZB\,(X,Y,Z)\,dY + C\,(X,Y,Z)\,dZ$ where by (1) $XA+YB+C=0$. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a rational map represented in the coordinates $(X,Y,Z) \in \mathbb{C}^3$ and $W \in \mathbb{C}^{n+1}$ by $\tilde{f} = (F_0, F_1, F_2^{\gamma})$ where F_0, F_1 and $F_2 \in \mathbb{C}[W]$ are homogeneous polynomials without common factors satisfying

$$deg(F_0) = deg(F_1) = \gamma.deg(F_2) = \nu.$$

The pull back foliation $f^*(\mathcal{G})$ is then defined by

$$\tilde{\eta}_{[f,\mathcal{G}]}\,(W) = \left[F_2\,(A \circ F)\,dF_0 + F_2\,(B \circ F)\,dF_1 + \gamma\,(C \circ F)\,dF_2\right],$$

where each coefficient of $\tilde{\eta}_{[f,\mathcal{G}]}\,(W)$ has degree $\Gamma = \nu \left[d + 1 + \frac{1}{\gamma}\right] - 1$. The crucial point here is that the mapping f sends the hypersurface $(F_2 = 0)$ contained in its critical set over the line invariant by $\mathcal{G}$.

Let $PB\,(\Gamma-1,\nu,\alpha,\gamma)$ be the closure in $\mathbb{F}\mathrm{ol}\,(\Gamma-1,n)$ of the set $\left\{\left[\tilde{\eta}_{[f,\mathcal{G}]}\right]\right\}$. It is an unirational irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\,(\Gamma-1,n)$. We will return to this point in Section 4. We observe that the arguments for the cases $\alpha = \beta = 1$ and $\alpha = \beta > 1$ are similar. Hence we can unify the two situations in a unique statement. The main result of this work is:

Theorem A. *$PB(\Gamma-1,\nu,\alpha,\gamma)$ is a unirational irreducible component of $\mathbb{F}\mathrm{ol}\,(\Gamma-1,n)$ for all $n \geq 3$, $deg(F_0).\alpha = deg(F_1).\alpha = deg(F_2).\gamma = \nu \geq 2$, such that $\alpha \geq 1$, $\gamma \geq 2$, $\nu \geq 2$ and $d \geq 2$ are integers.*

2. Branched rational maps

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a rational map and $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^3$ is it natural lifting in homogeneous coordinates. The *indeterminacy locus* of f is, by definition, the set $I\,(f) = \Pi_n\left(\tilde{f}^{-1}\,(0)\right)$. We characterize the set of rational maps used throughout this text as follows:

Definition 2.1. We denote by $BRM\,(n,\nu,\alpha,\gamma)$ the set of maps $\left\{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2\right\}$ of degree ν given by $f = (F_0^{\alpha} : F_1^{\alpha} : F_2^{\gamma})$ where F_0, F_1 and F_2 are homogeneous polynomials without common factors, with $deg\,(F_0)\,.\alpha = deg\,(F_1)\,.\alpha = deg\,(F_2)\,.\gamma = \nu$, where $\nu \geq 2$, $\alpha \geq 1$ and $\gamma \geq 2$ are integers.

Let us fix some coordinates $(z_0, ..., z_n)$ on $\mathbb{C}^{n+1}$ and (X, Y, Z) on $\mathbb{C}^3$ and denote by $(F_0^\alpha, F_1^\alpha, F_2^\gamma)$ the components of f relative to these coordinates. Let us note that the indeterminacy locus $I(f)$ is the intersection of the three hypersurfaces $(F_0 = 0)$, $(F_1 = 0)$ and $(F_2 = 0)$.

Definition 2.2. We say that $f \in BRM\,(n, \nu, \alpha, \gamma)$ is *generic* if for all $p \in \tilde{f}^{-1}(0) \setminus \{0\}$ we have $dF_0\,(p) \wedge dF_1\,(p) \wedge dF_2\,(p) \neq 0$.

This is equivalent to saying that $f \in BRM\,(n, \nu, \alpha, \gamma)$ is *generic* if $I(f)$ is the transverse intersection of the 3 hypersurfaces $(F_0 = 0)$, $(F_1 = 0)$ and $(F_2 = 0)$. As a consequence we have that the set $I(f)$ is smooth. For instance, if $n = 3$, f is generic and $deg(f) = \nu$, then by Bezout's theorem $I\,(f)$ consists of $\frac{\nu^3}{\alpha^2\gamma}$ distinct points with multiplicity $\alpha^2\gamma$. If $n = 4$, then $I\,(f)$ is a smooth connected algebraic curve in $\mathbb{P}^4$ of degree $\frac{\nu^3}{\alpha^2\gamma}$. In general, for $n \geq 4$, $I\,(f)$ is a smooth connected algebraic submanifold of $\mathbb{P}^n$ of degree $\frac{\nu^3}{\alpha^2\gamma}$ and codimension three.

Denote $\nabla F_k = (\frac{\partial F_k}{\partial z_0}, ..., \frac{\partial F_k}{\partial z_n})$. Consider the derivative matrix

$$M = \begin{bmatrix} \alpha \left(F_0^{\alpha-1}\right) \nabla F_0 \\ \alpha \left(F_1^{\alpha-1}\right) \nabla F_1 \\ \gamma \left(F_2^{\gamma-1}\right) \nabla F_2 \end{bmatrix}.$$

The critical set of $\tilde{f}$ is given by the points of $\mathbb{C}^{n+1} \setminus 0$ where rank$(M) \leq 3$; it is the union of two sets. The first is given by the set of $\left\{P \in \mathbb{C}^{n+1} \setminus 0\right\} = X_1$ such that the rank of the following matrix

$$N = \begin{bmatrix} \nabla F_0 \\ \nabla F_1 \\ \nabla F_2 \end{bmatrix}$$

is smaller than 3. The second is the subset

$$X_2 = \left\{P \in \mathbb{C}^{n+1} \setminus \{0\} \,|\, \left(F_0^{\alpha-1}\right)\left(F_1^{\alpha-1}\right)\left(F_2^{\gamma-1}\right)(P) = 0\right\}.$$

Denote $P\,(f) = \Pi_n\,(X_1 \cup X_2)$. The set of generic maps will be denoted by $Gen\,(n, \nu, \alpha, \gamma)$. We state the following result whose proof is standard in algebraic geometry:

Proposition 2.3. *$Gen\,(n, \nu, \alpha, \gamma)$ is a Zariski dense subset of $BRM\,(n, \nu, \alpha, \gamma)$.*

Once the case of foliations which are pull-backs of three invariant straight have been already discussed in [CS]. We will concentrate only on the case where $\alpha = 1$. The case $\alpha > 1$ is obtained following the same ideas.

3. Foliations with one invariant line

3.1. **Basic facts.** Denote by $I_1(d, 2)$ the set of the holomorphic foliations on $\mathbb{P}^2$ of degree $d \geq 2$ that leaves the line $Z = 0$ invariant. We observe that any foliation which has 1 invariant straight line can be carried to one of these by a linear automorphism of $\mathbb{P}^2$. The relation $XA + YB + C = 0$ enables to parametrize $I(d, 2)$ as follows

$$\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))^{\times 2} \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))^{\times 3}$$
$$(A, B) \mapsto (A, B, -XA - YB).$$

We let the group of linear automorphisms of $\mathbb{P}^2$ act on $I_1(d, 2)$. After this procedure we obtain a set of foliations of degree d that we denote by $Il_1(d, 2)$.

We are interested in making deformations of foliations and for our purposes we need a subset of $Il_1(d,2)$ with good properties (foliations having few algebraic invariant curves and only hyperbolic singularities). We explain this properties in detail. Let $q \in U$ be an isolated singularity of a foliation $\mathcal{G}$ defined on an open subset of $U \subset \mathbb{C}^2$. We say that q is *nondegenerate* if there exists a holomorphic vector field X tangent to $\mathcal{G}$ in a neighborhood of q such that $DX(q)$ is nonsingular. In particular q is an isolated singularity of X. Let q be a nondegenerate singularity of $\mathcal{G}$. The *characteristic numbers* of q are the quotients λ and λ^{-1} of the eingenvalues of $DX(q)$, which do not depend on the vector field X chosen. If $\lambda \notin \mathbb{Q}_+$ then $\mathcal{G}$ exhibits exactly two (smooth and transverse) *local separatrices* at q, S_q^+ and S_q^- with eigenvalues λ_q^+ and λ_q^- and which are tangent to the characteristic directions of a vector field X. The characteristic numbers (also called Camacho-Sad index) of these local separatrices are given by

$$I(\mathcal{G}, S_q^+) = \frac{\lambda_q^-}{\lambda_q^+} \text{ and } I(\mathcal{G}, S_q^-) = \frac{\lambda_q^+}{\lambda_q^-}.$$

The singularity is *hyperbolic* if the characteristic numbers are nonreal. We introduce the following spaces of foliations:

(1) $ND(d,2) = \{\mathcal{G} \in \mathbb{F}\mathrm{ol}(d,2);$ the singularities of $\mathcal{G}$ are nondegenerate$\}$,
(2) $\mathcal{H}(d,2) = \{\mathcal{G} \in ND(d,2);$ any characteristic number λ of $\mathcal{G}$ satisfies $\lambda \in \mathbb{C}\backslash\mathbb{R}\}$.

It is a well-known fact [LN2] that $\mathcal{H}(d,2)$ contains an open and dense subset of $\mathbb{F}\mathrm{ol}(d,2)$. Denote by $A(d) = Il_1(d,2) \cap \mathcal{H}(d,2)$. Observe that $A(d)$ is a Zariski dense subset of $Il_1(d,2)$. Concerning the set $ND(d,2)$, we have the following result, proved in [LN2].

Proposition 3.1. *Let $\mathcal{G}_0 \in ND(d,2)$. Then $\#Sing(\mathcal{G}_0) = d^2 + d + 1 = N(d)$. Moreover if $Sing(\mathcal{G}_0) = \{p_1^0, ..., p_N^0\}$ where $p_i^0 \neq p_j^0$ if $i \neq j$, then there are connected neighborhoods $U_j \ni p_j$, pairwise disjoint, and holomorphic maps $\phi_j : \mathcal{U} \subset ND(d,2) \to U_j$, where $\mathcal{U} \ni \mathcal{G}_0$ is an open neighborhood, such that for $\mathcal{G} \in \mathcal{U}$, $(Sing(\mathcal{G}) \cap U_j) = \phi_j(\mathcal{G})$ is a nondegenerate singularity. In particular, $ND(d,2)$ is open in $\mathbb{F}\mathrm{ol}(2,d)$. Moreover, if $\mathcal{G}_0 \in \mathcal{H}(d,2)$ then the two local separatrices as well as their associated eigenvalues depend analytically on $\mathcal{G}$.*

In the paper [LN.S.Sc] which is related to the topological rigidity of foliations on $\mathbb{P}^2$ in the spirit of Ilyashenko's works. The authors have proved the following useful result see[LN.S.Sc, Theorem 3, p.385].

Theorem 3.2. *Let $d \geq 2$. There exists an non empty open and dense subset $M(d) \subset A(d)$, such that if $\mathcal{G} \in M(d)$ then the only algebraic invariant curve of $\mathcal{G}$ is the line.*

4. Ramified pull-back components - Generic conditions

Let us fix a coordinate system (X, Y, Z) on $\mathbb{P}^2$ and denote by ℓ the straight line that corresponds to the plane $Z = 0$ in $\mathbb{C}^3$, respectively. Let us denote by $\tilde{M}(d)$ the subset $M(d) \cap I_1(d,2)$.

Definition 4.1. Let $f \in Gen\,(n, \nu, 1, \gamma)$. We say that $\mathcal{G} \in M\,(d)$ is in generic position with respect to f if $[Sing\,(\mathcal{G}) \cap Y_2] = \emptyset$, where

$$Y_2(f) = Y_2 := \Pi_2 \left[\tilde{f} \left\{ w \in \mathbb{C}^{n+1} | dF_0\,(w) \wedge dF_1\,(w) \wedge dF_2\,(w) = 0 \right\} \right]$$

and ℓ is $\mathcal{G}$-invariant.

In this case we say that $(f, \mathcal{G})$ is a generic pair. In particular, when we fix a map $f \in Gen(n, \nu, 1, \gamma)$ the set $\mathcal{A} = \{\mathcal{G} \in M\,(d)\,| Sing\,(\mathcal{G}) \cap Y_2(f) = \emptyset\}$ is an open and dense subset in $M(d)$ [LN.Sc], since $VC(f)$ is an algebraic curve in $\mathbb{P}^2$. The set $U_1 := \{(f, \mathcal{G}) \in Gen(n, \nu, 1, \gamma) \times \tilde{M}\,(d)\,| Sing\,(\mathcal{G}) \cap Y_2(f) = \emptyset\}$ is an open and dense subset of $Gen(n, \nu, 1, \gamma) \times \tilde{M}\,(d)$. Hence the set $\mathcal{W} := \left\{ \tilde{\eta}_{[f,\mathcal{G}]} |\, (f, \mathcal{G}) \in U_1 \right\}$ is an open and dense subset of $PB\,(\Gamma - 1, \nu, 1, \gamma)$.

Proposition 4.2. *If $\mathcal{F}$ comes from a generic pair, then the degree of $\mathcal{F}$ is*

$$\nu \left[d + 1 + \frac{1}{\gamma} \right] - 2.$$

The proof of this fact can be obtained as in the case treated in [CS].

Consider the set of foliations $Il_1\,(d, 2)$, $d \geq 2$, and the following map:

$$\begin{array}{rcl} \Phi : BRM\,(n, \nu, 1, \gamma) \times Il_1\,(d, 2) & \to & \mathbb{F}\mathrm{ol}\,(\Gamma - 1, n) \\ (f, \mathcal{G}) & \to & f^*\,(\mathcal{G}) = \Phi\,(f, \mathcal{G})\,. \end{array}$$

The image of Φ can be written as:

$$\Phi\,(f, \mathcal{G}) = [F_2\,(A \circ F)\,dF_0 + F_2\,(B \circ F)\,dF_1 + \gamma\,(C \circ F)\,dF_2]\,.$$

Recall that $\Phi\,(f, \mathcal{G}) = \tilde{\eta}_{[f,\mathcal{G}]}$. More precisely, let $PB(\Gamma - 1, n, \nu, 1, 1, \gamma)$ be the closure in $\mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$ of the set of foliations $\mathcal{F}$ of the form $f^*\,(\mathcal{G})$, where $f \in BRM\,(n, \nu, 1, \gamma)$ and $\mathcal{G} \in Il_1(2, d)$. Since $BRM\,(n, \nu, 1, \gamma)$ and $Il_1(2, d)$ are irreducible algebraic sets and the map $(f, \mathcal{G}) \to f^*\,(\mathcal{G}) \in \mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$ is an algebraic parametrization of $PB(\Gamma - 1, \nu, 1, \gamma)$, we have that $PB(\Gamma - 1, \nu, 1, \gamma)$ is an irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$. Moreover, the set of generic pull-back foliations $\{\mathcal{F}; \mathcal{F} = f^*(\mathcal{G}),$ where $(f, \mathcal{G})$ is a generic pair$\}$ is an open (not Zariski) and dense subset of $PB(\Gamma - 1, \nu, 1, \gamma)$ for $\gamma \geq 2 \in \mathbb{N}$, $\nu \geq 2 \in \mathbb{N}$ and $d \geq 2 \in \mathbb{N}$.

5. Description of generic ramified pull-back foliations on $\mathbb{P}^n$

5.1. **The Kupka set.** Let τ be a singularity of $\mathcal{G}$ and $V_\tau = \overline{f^{-1}(\tau)}$. If $(f, \mathcal{G})$ is a generic pair then $V_\tau \backslash I(f)$ is contained in the Kupka set of $\mathcal{F}$. As an example we detail the case where τ is a singularity over the invariant line, say $\tau = [1:0:0]$. Fix $p \in V_\tau \backslash I(f)$. There exist local analytic coordinate systems such that $f(x, y, z) = (x, y^\gamma) = (u, v)$. Suppose that $\mathcal{G}$ is represented by the 1-form ω; the hypothesis of $\mathcal{G}$ being of Hyperbolic-type implies that we can suppose $\omega(u, v) = \lambda_1 u(1 + R(u, v))dv - \lambda_2 v du$, where $\frac{\lambda_2}{\lambda_1} \in \mathbb{C} \backslash \mathbb{R}$. We obtain $\tilde{\omega}(x, y) = f^*(\omega) = (y^{\gamma-1})(\lambda_1 \gamma x(1 + R(x, y^\gamma)dy - \lambda_2 y dx) = (y^{\gamma-1})\hat{\omega}(x, y)$ and so $d\hat{\omega}(p) \neq 0$. Therefore if p is as before it belongs to the Kupka-set of $\mathcal{F}$. For the other points the argumentation is analogous. This is the well known Kupka-Reeb phenomenon, and we say that p is contained in the Kupka-set of $\mathcal{F}$. It is known that this local product structure is stable under small perturbations of $\mathcal{F}$ for instance, see [K],[G.LN].

5.2. **Generalized Kupka and quasi-homogeneous singularities.** In this section we will recall the quasi-homogeneous singularities of an integrable holomorphic 1-form. They appear in the indeterminacy set of f and play a central role in great part of the proof of Theorem B.

Definition 5.1. Let ω be an holomorphic integrable 1-form defined in a neighborhood of $p \in \mathbb{C}^3$. We say that p is a Generalized Kupka(GK) singularity of ω if $\omega(p) = 0$ and either $d\omega(p) \neq 0$ or p is an isolated zero of $d\omega$.

Let ω be an integrable 1-form in a neighborhood of $p \in \mathbb{C}^3$ and μ be a holomorphic 3-form such that $\mu(p) \neq 0$. Then $d\omega = i_{\mathcal{Z}}(\mu)$ where $\mathcal{Z}$ is a holomorphic vector field.

Definition 5.2. We say that p is a quasi-homogeneous singularity of ω if p is an isolated singularity of $\mathcal{Z}$ and the germ of $\mathcal{Z}$ at p is nilpotent, that is, if $L = D\mathcal{Z}(p)$ then all eigenvalues of L are equals to zero.

This definition is justified by the following result that can be found in [LN2] or [C.CA.G.LN]:

Theorem 5.3. *Let p be a quasi-homogeneous singularity of an holomorphic integrable 1-form ω. Then there exists two holomorphic vector fields S and $\mathcal{Z}$ and a local chart $U := (x_0, x_1, x_2)$ around p such that $x_0(p) = x_1(p) = x_2(p) = 0$ and:*

(a) $\omega = \lambda i_S i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$, $\lambda \in \mathbb{Q}_+$ $d\omega = i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$ *and* $\mathcal{Z} = (rot(\omega))$;
(b) $S = p_0 x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2}$, *where, p_0, p_1, p_2 are positive integers with $g.c.d(p_0, p_1, p_2) = 1$;*
(c) *p is an isolated singularity for $\mathcal{Z}$, $\mathcal{Z}$ is polynomial in the chart $U := (x_0, x_1, x_2)$ and $[S, \mathcal{Z}] = \ell\mathcal{Z}$, where $\ell \geq 1$.*

Definition 5.4. Let p be a quasi-homogeneous singularity of ω. We say that it is of the type $(p_0 : p_1 : p_2; \ell)$, if for some local chart and vector fields S and $\mathcal{Z}$ the properties $(a), (b)$ and (c) of the Theorem 5.3 are satisfied.

We can now state the stability result, whose proof can be found in [C.CA.G.LN]:

Proposition 5.5. *Let $(\omega_s)_{s\in\Sigma}$ be a holomorphic family of integrable 1-forms defined in a neighborhood of a compact ball $B = \{z \in \mathbb{C}^3; |z| \leq \rho\}$, where Σ is a neighborhood of $0 \in \mathbb{C}^k$. Suppose that all singularities of ω_0 in B are GK and that $sing(d\omega_0) \subset int(B)$. Then there exists $\epsilon > 0$ such that if $s \in B(0, \epsilon) \subset \Sigma$, then all singularities of ω_s in B are GK. Moreover, if $0 \in B$ is a quasi-homogeneous singularity of type $(p_0 : p_1 : p_2; \ell)$ then there exists a holomorphic map $B(0, \epsilon) \ni s \mapsto z(s)$, such that $z(0) = 0$ and $z(s)$ is a GK singularity of ω_s of the same type (quasi-homogeneous of the type $(p_0 : p_1 : p_2; \ell)$, according to the case).*

Let us describe $\mathcal{F} = f^*(\mathcal{G})$ in a neighborhood of a point $p \in I(f)$. It is easy to show that there exists a local chart $(U, (x_0, x_1, x_2, y) \in \mathbb{C}^3 \times \mathbb{C}^{n-2})$ around p such that the lifting $\tilde{f}$ of f is of the form $\tilde{f}|_U = (x_0, x_1, x_2^\gamma) : U \to \mathbb{C}^3$. In particular $\mathcal{F}|_{U(p)}$ is represented by the 1-form

$$\eta(x_0, x_1, x_2, y) = x_2.A(x_0, x_1, x_2^\gamma)dx_0 + x_2.B(x_0, x_1, x_2^\gamma)dx_1 + \gamma C(x_0, x_1, x_2^\gamma)dx_2. \tag{5.1}$$

Let us now obtain the vector field S as in Theorem 5.3. Consider the radial vector field $R = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$. Note that in the coordinate system above it transforms into

$$x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \frac{1}{\gamma}x_2\frac{\partial}{\partial x_2}.$$

Since the eigenvalues of S have to be integers, after a multiplication by γ we obtain

$$S = \gamma x_0\frac{\partial}{\partial x_0} + \gamma x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}.$$

Let us concentrate in the case $n = 3$.

Lemma 5.6. *If η and S are as above then we have $L_S\eta = [1 + \gamma(1 + d)]\eta$.*

Proof. We just have to use Cartan's formula for the Lie's derivative, $L_S\eta = i_S d\eta + d(i_S\eta)$. The details are left for the reader. □

Lemma 5.7. *If $p \in I(f)$ then p is a quasi-homogeneous singularity of η.*

Proof. First of all note that $i_S\eta = 0$. From the computations obtained in lemma 5.6, we have that $L_S\eta = m\eta$, where $m = [1 + \gamma(1 + d)]$. This implies that the singular set of η is invariant under the flow of S. The vector field $\mathcal{Z}$ such that $\eta = i_S i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$ is given by

$$\mathcal{Z} = \mathcal{Z}_0(x_0, x_1, x_2)\frac{\partial}{\partial x_0} + \mathcal{Z}_1(x_0, x_1, x_2)\frac{\partial}{\partial x_1} + \mathcal{Z}_2(x_0, x_1, x_2)\frac{\partial}{\partial x_2}$$

where for $i = 0, 1$ we have $\mathcal{Z}_i(x_0, x_1, x_2) = \tilde{A}_i(x_0, x_1, x_2^{\gamma})$ and $\mathcal{Z}_2(x_0, x_1, x_2) = x_2.\tilde{A}_2(x_0, x_1, x_2^{\gamma})$ moreover for $i = 0, 1$ the polynomials $\tilde{A}_i(0, 0, 0) = 0$ and $\tilde{A}_2(0, 0, 0) = 0$. We observe that these polynomials are not unique. On the other hand, they have to satisfy the following relations:

$$\begin{aligned}
A(x_0, x_1, x_2^{\gamma}) &= \gamma x_1\tilde{A}_2(x_0, x_1, x_2^{\gamma}) - \tilde{A}_1(x_0, x_1, x_2^{\gamma})\\
B(x_0, x_1, x_2^{\gamma}) &= \tilde{A}_0(x_0, x_1, x_2^{\gamma}) - \gamma x_0\tilde{A}_2(x_0, x_1, x_2^{\gamma})\\
C(x_0, x_1, x_2^{\gamma}) &= x_0\tilde{A}_1(x_0, x_1, x_2^{\gamma}) - x_1\tilde{A}_1(x_0, x_1, x_2^{\gamma})
\end{aligned}$$

We must show that the origin is an isolated singularity of $\mathcal{Z}$ and all eigenvalues of $D\mathcal{Z}(0)$ are 0. By straightforward computation we find that the Jacobian matrix $D\mathcal{Z}(0)$ is the null matrix, hence all its eigenvalues are null. Since all singular curves of $\mathcal{F}$ in a neighborhood $(U, (x_0, x_1, x_2))$ of 0 are of Kupka type, as proved in Section 5.1, it follows that the origin is an isolated sigularity of $\mathcal{Z}$. Note that the unique singularities of η in the neighborhood $(U, (x_0, x_1, x_2))$ of 0 come from $\tilde{f}^*Sing(\mathcal{G})$; this follows from the fact that $Sing\,(\mathcal{G}) \cap (VC(f)\backslash\ell) = \emptyset$. On the other hand we have seen that $(f)^{-1}(sing(\mathcal{G}))\backslash I(f)$ is contained in the Kupka set of $\mathcal{F}$. Hence the point p is an isolated singularity of $d\eta$ and thus an isolated singularity of $\mathcal{Z}$. □

As a consequence, in the case $n = 3$ any $p \in I(f)$ is a quasi-homogeneous singularity of type $[\gamma : \gamma : 1]$. In the case $n \geq 4$ the argument is analogous. Moreover, in this case there will be a local structure product near any point $p \in I(f)$. In fact in the case $n \geq 4$ we have:

Corollary 5.8. *Let $(f, \mathcal{G})$ be a generic pair. Let $p \in I(f)$ and η an 1-form defining $\mathcal{F}$ in a neighborhood of p. Then there exists a 3-plane $\Pi \subset \mathbb{C}^n$ such that $d(\eta)|_{\Pi}$ has an isolated singularity at $0 \in \Pi$.*

Proof. Immediate from the local product structure. □

5.3. **Deformations of the singular set.** In this section we give some auxiliary lemmas which assist in the proof of Theorem A. We have constructed an open and dense subset $\mathcal{W}$ inside $PB(\Gamma-1,\nu,1,1,\gamma)$ containing the generic pull-back foliations. We will show that for any foliation $\mathcal{F}\in\mathcal{W}$ and any germ of a holomorphic family of foliations $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}$ such that $\mathcal{F}_0=\mathcal{F}$ we have $\mathcal{F}_t\in PB(\Gamma-1,\nu,1,1,\gamma)$ for all $t\in(\mathbb{C},0)$.

Lemma 5.9. *There exists a germ of isotopy of class C^∞, $(I(t))_{t\in(\mathbb{C},0)}$ having the following properties:*

(i) *$I(0)=I(f_0)$ and $I(t)$ is algebraic and smooth of codimension 3 for all $t\in(\mathbb{C},0)$.*

(ii) *For all $p\in I(t)$, there exists a neighborhood $U(p,t)=U$ of p such that $\mathcal{F}_t$ is equivalent to the product of a regular foliation of codimension 3 and a singular foliation $\mathcal{F}_{p,t}$ of codimension one given by the 1-form $\eta_{p,t}$.*

Remark 5.10. *The family of 1-forms $\eta_{p,t}$, represents the quasi-homogeneous foliation given by the Proposition 5.5.*

Proof. See [LN0, lema 2.3.2, p.81]. □

Remark 5.11. *In the case $n>3$, the variety $I(t)$ is connected since $I(f_0)$ is connected. The local product structure in $I(t)$ implies that the transversal type of $\mathcal{F}_t$ is constant. In particular, $\mathcal{F}_{p,t}$, does not depend on $p\in I(t)$. In the case $n=3$, $I(t)=p_1(t),...,p_j(t),...,p_{\frac{\nu^3}{\gamma}}(t)$ and we can not guarantee a priori that $\mathcal{F}_{p_i,t}=\mathcal{F}_{p_j,t}$, if $i\neq j$.*

The singular set of $\mathcal{G}_0$ can be divided in two subsets $\mathcal{S}_W(\mathcal{G}_0)$, $\mathcal{S}_\ell(\mathcal{G}_0)$. We know that $\#\mathcal{S}_W(\mathcal{G}_0)=d^2$, $\#\mathcal{S}_\ell(\mathcal{G}_0)=(d+1)$. Let $\tau\in Sing(\mathcal{G}_0)$ and $K(\mathcal{F}_0)=\cup_{\tau\in Sing(\mathcal{G}_0)}V_\tau\backslash I(f_0)$ where $V_\tau=\overline{f_0^{-1}(\tau)}$. As in Lemma 5.9, let us consider a representative of the germ $(\mathcal{F}_t)_t$, defined on a disc $D_\delta:=(|t|<\delta)$.

Lemma 5.12. *There exist $\epsilon>0$ and smooth isotopies $\phi_\tau: D_\epsilon\times V_\tau\to\mathbb{P}^n, \tau\in Sing(\mathcal{G}_0)$, such that $V_\tau(t)=\phi_\tau(\{t\}\times V_\tau)$ satisfies:*

(a) *$V_\tau(t)$ is an algebraic subvariety of codimension two of $\mathbb{P}^n$ and $V_\tau(0)=V_\tau$ for all $\tau\in Sing(\mathcal{G}_0)$ and for all $t\in D_\epsilon$.*

(b) *$I(t)\subset V_\tau(t)$ for all $\tau\in Sing(\mathcal{G}_0)$ and for all $t\in D_\epsilon$. Moreover, if $\tau\neq\tau'$, and $\tau,\tau'\in Sing(\mathcal{G}_0)$, we have $V_\tau(t)\cap V_{\tau'}(t)=I(t)$ for all $t\in D_\epsilon$ and the intersection is transversal.*

(c) *$V_\tau(t)\backslash I(t)$ is contained in the Kupka-set of $\mathcal{F}_t$ for all $\tau\in Sing(\mathcal{G}_0)$ and for all $t\in D_\epsilon$. In particular, the transversal type of $\mathcal{F}_t$ is constant along $V_\tau(t)\backslash I(t)$.*

Proof. See [LN0, lema 2.3.3, p.83]. □

6. Proof of theorem A

6.1. **End of the proof of Theorem A.** We divide the end of the proof of Theorem A in two parts. In the first part we construct a family of rational maps $f_t:\mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $f_t\in Gen(n,\nu,1,\gamma)$, such that $(f_t)_{t\in D_\epsilon}$ is a deformation of f_0 and the subvarieties V_τ, $\tau\in Sing(\mathcal{G}_0)$, are fibers of f_t for all t. In the second part we show that there exists a family of foliations $(\mathcal{G}_t)_{t\in D_\epsilon}, \mathcal{G}_t\in\mathcal{A}$ (see Section 4) such that $\mathcal{F}_t=f_t^*(\mathcal{G}_t)$ for all $t\in D_\epsilon$.

6.1.1. *Part 1.* Let us define the family of candidates that will be a deformation of the mapping f_0. Set $V_a = \overline{f_0^{-1}(a)}$, $V_b = \overline{f_0^{-1}(b)}$, $V_c = \overline{f_0^{-1}(c)}$, where $a = [0 : 0 : 1]$, $b = [0 : 1 : 0]$ and $c = [1 : 0 : 0]$ and denote by $V_{\tau^*} = \overline{f_0^{-1}(\tau^*)}$, where $\tau^* \in Sing(\mathcal{G}_0)\backslash\{a, b, c\}$. In this coordinate system the points b and c belong to ℓ.

Proposition 6.1. *Let $(\mathcal{F}_t)_{t\in D_\epsilon}$ be a deformation of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$, where $(f_0, \mathcal{G}_0)$ is a generic pair, with $\mathcal{G}_0 \in \mathcal{A}$, $f_0 \in Gen\,(n, \nu, 1, \gamma)$ and $deg(f_0) = \nu \geq 2$. Then there exists a deformation $(f_t)_{t\in D_\epsilon}$ of f_0 in $Gen\,(n, \nu, 1, \gamma)$ such that:*

(i) *$V_a(t), V_b(t)$ and $V_c(t)$ are fibers of $(f_t)_{t\in D_{\epsilon'}}$.*
(ii) *$I(t) = I(f_t), \forall t \in D_{\epsilon'}$.*

Proof. Let $\tilde{f}_0 = (F_0, F_1, F_2^\gamma) : \mathbb{C}^{n+1} \to \mathbb{C}^3$ be the homogeneous expression of f_0. Then V_c, V_b, and V_a appear as the complete intersections $(F_1 = F_2 = 0)$, $(F_0 = F_2 = 0)$, and $(F_0 = F_1 = 0)$ respectively. Hence $I(f_0) = V_a \cap V_b = V_a \cap V_c = V_b \cap V_c$. It follows from [Ser, section 4.6, p.235-236] that $V_a(t)$ is a complete intersection, say $V_a(t) = (F_0(t) = F_1(t) = 0)$, where $(F_0(t))_{t\in D_{\epsilon'}}$ and $(F_1(t))_{t\in D_{\epsilon'}}$ are deformations of F_0 and F_1 and $D_{\epsilon'}$ is a possibly smaller neighborhood of 0. Moreover, $F_0(t) = 0$ and $F_1(t) = 0$ meet transversely along $V_a(t)$. In the same way, it is possible to define $V_c(t)$ and $V_b(t)$ as complete intersections, say $(\hat{F}_1(t) = F_2(t) = 0)$ and $(\hat{F}_0(t) = \hat{F}_2(t) = 0)$ respectively, where $(F_j(t))_{t\in D_{\epsilon'}}$ and $(\hat{F}_j(t))_{t\in D_{\epsilon'}}$ are deformations of F_j, $0 \leq j \leq 2$.

We will prove that we can find polynomials $P_0(t)$, $P_1(t)$ and $P_2(t)$ such that $V_c(t) = (P_1(t) = P_2(t) = 0)$, $V_b(t) = (P_0(t) = P_2(t) = 0)$ and $V_a(t) = (P_0(t) = P_1(t) = 0)$. Observe first that since $F_0(t), F_1(t)$ and $F_2(t)$ are near F_0, F_1 and F_2 respectively, they meet as a regular complete intersection at:

$$J(t) = (F_0(t) = F_1(t) = F_2(t) = 0) = V_a(t) \cap (F_2(t) = 0).$$

Hence $J(t) \cap (\hat{F}_1(t) = 0) = V_c(t) \cap V_a(t) = I(t)$, which implies that $I(t) \subset J(t)$. Since $I(t)$ and $J(t)$ have $\frac{\nu^3}{\gamma}$ points, we have that $I(t) = J(t)$ for all $t \in D_{\epsilon'}$.

Remark 6.2. *In the case $n \geq 4$, both sets are codimension-three smooth and connected submanifolds of $\mathbb{P}^n$, implying again that $I(t) = J(t)$. In particular, we obtain that*

$$I(t) = (F_0(t) = F_1(t) = F_2(t) = 0) \subset (\hat{F}_j(t) = 0), 0 \leq j \leq 2.$$

We will use the following version of Noether's Normalization Theorem (see [LN0] p 86):

Lemma 6.3. *(Noether's Theorem) Let $G_0, ..., G_k \in \mathbb{C}[z_1, ..., z_m]$ be homogeneous polynomials where $0 \leq k \leq m$ and $m \geq 2$, and $X = (G_0 = ... = G_k = 0)$. Suppose that the set $Y := \{p \in X | dG_0(p) \wedge ... \wedge dG_k(p) = 0\}$ is either 0 or $\emptyset$. If $G \in \mathbb{C}[z_1, ..., z_m]$ satisfies $G|_X \equiv 0$, then $G \in < G_0, ..., G_k >$.*

Take $k = 2$, $G_0 = F_0(t)$, $G_1 = F_1(t)$ and $G_2 = F_2(t)$. Using Noether's Theorem with $Y = 0$ and the fact that all polynomials involved are homogeneous, we have $\hat{F}_1(t) \in < F_0(t), F_1(t), F_2(t) >$. Since $deg(F_0(t)) = deg(F_1(t)) > deg(F_2(t))$, we conclude that $\hat{F}_1(t) = F_1(t) + g(t)F_2(t)$, where $g(t)$ is a homogeneous polynomial of degree $deg(F_1(t)) - deg(F_2(t))$. Moreover observe that $V_c(t) = V(\hat{F}_1(t), F_2(t)) = V(F_1(t), F_2(t))$, where $V(H_1, H_2)$ denotes the projective algebraic variety defined by $(H_1 = H_2 = 0)$. Similarly for $V_b(t)$ we have that $\hat{F}_2(t) \in < F_0(t), F_1(t), F_2(t) >$. On the other hand, since $\hat{F}_2(t)$ has the lowest degree, we can assume that $\hat{F}_2(t) = F_2(t)$.

In an analogous way we have that $\hat{F}_0(t) = F_0(t) + m(t)F_1(t) + n(t)F_2(t)$ for the polynomial $\hat{F}_0(t)$. Now observe that $V(\hat{F}_0(t), \hat{F}_2(t)) = V(F_0(t) + m(t)F_1(t), F_2(t))$ where $m(t) \in \mathbb{C}$ satisfying $m(0) = 0$. Hence we can define the family of polynomials as being $P_0(t) = F_0(t) + m(t)F_1(t)$, $P_1(t) = F_1(t)$ and $P_2(t) = F_2(t)$. This defines a family of mappings $(f_t)_{t\in D_{\epsilon'}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, and $V_a(t)$, $V_b(t)$ and $V_c(t)$ are fibers of f_t for fixed t. Observe that, for ϵ' sufficiently small, $(f_t)_{t\in D_{\epsilon'}}$ is generic in the sense of definition 3.2, and its indeterminacy locus $I(f_t)$ is precisely $I(t)$. Moreover, since $Gen(3, \nu, 1, \gamma)$ is open, we can suppose that this family $(f_t)_{t\in D_{\epsilon'}}$ is in $Gen\,(3, \nu, 1, \gamma)$. This concludes the proof of proposition 5.10. $\square$

We observe that this family can be considered also as a family of mappings $(\overline{f}_t)_{t\in D_{\epsilon'}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2_{[\gamma,\gamma,1]}$, where $\overline{f}_t = (P_0(t), F_1(t), F_2(t))$ where $\mathbb{P}^2_{[\gamma,\gamma,1]}$ denotes the weighted projective plane with weights $(\gamma, \gamma, 1)$. Moreover, using the map

$$\begin{array}{rcl} f_w : \mathbb{P}^2_{[\gamma,\gamma,1]} & \to & \mathbb{P}^2 \\ (x_0 : x_1 : x_2) & \to & (x_0 : x_1 : x_2^{\gamma}) \end{array}$$

we can factorize f_t as being $f_t = f_w \circ \overline{f_t}$ as shown in the diagram below:

$$\begin{tikzcd} \mathbb{P}^3 \arrow[rr, "f_t"] \arrow[dr, "\overline{f_t}"'] & & \mathbb{P}^2 \\ & \mathbb{P}_{[\gamma,\gamma,1]} \arrow[ur, "f_w"'] & \end{tikzcd}$$

Now we will prove that the remaining curves $V_\tau(t)$ are also fibers of f_t. In the local coordinates $X(t) = (x_0(t), x_1(t), x_2(t))$ near some point of $I(t)$ we have that the vector field S is diagonal and the components of the map f_t are written as follows:

$$\begin{aligned} P_0(t) &= u_{0t}x_0(t) + x_1(t)x_2(t)h_{0t} \\ P_1(t) &= u_{1t}x_1(t) + x_0(t)x_2(t)h_{1t} \\ P_2(t) &= u_{2t}x_2(t) + x_0(t)x_1(t)h_{2t} \end{aligned} \tag{6.1}$$

where the functions $u_{it} \in \mathcal{O}^*(\mathbb{C}^3, 0)$ and $h_{it} \in \mathcal{O}(\mathbb{C}^3, 0), 0 \le i \le 2$. Note that when the parameter t goes to 0 the functions $h_i(t), 0 \le i \le 2$ also goes to 0. We want to show that an orbit of the vector field S in the coordinate system $X(t)$ that extends globally like a singular curve of the foliation $\mathcal{F}_t$ is a fiber of f_t.

Lemma 6.4. *Any generic orbit of the vector field S that extends globally as singular curve of the foliations $\mathcal{F}_t$ is also a fiber of f_t for fixed t.*

Proof. To simplify the notation we will omit the index t. Let $\delta(s)$ be a generic orbit of the vector field S (here by a generic orbit we mean an orbit that is not a coordinate axis). We can parametrize $\delta(s)$ as $s \to (as^\gamma, bs^\gamma, cs)$, $a \neq 0, b \neq 0, c \neq 0$. Without loss of generality we can suppose that $a = b = c = 1$. We have

$$f_t(\delta(s)) = [(s^\gamma u_0 + s^{(1+\gamma)}h_0) : (s^\gamma u_1 + s^{(1+\gamma)}h_1) : (su_2 + s^{2\gamma}h_2)^\gamma].$$

Hence we can extract the factor s^γ from $f_t(\delta(s))$ and we obtain

$$f_t(\delta(s)) = [(u_0 + sh_0) : (u_1 + s^l h_1) : (u_2 + s^{2\gamma}h_2)^\gamma]. \tag{6.2}$$

Since V_τ is a fiber, $f_0(V_\tau) = [d : e : f] \in \mathbb{P}^2$ with $d \neq 0, e \neq 0, f \neq 0$. If we take a covering of $I(f) = \{p_1, ..., p_{\frac{\nu^3}{\gamma}}\}$ by small open balls $B_j(p_j)$, $1 \le j \le \frac{\nu^3}{\gamma}$,

the set $V_\tau \setminus \cup_j B_j(p_j)$ is compact. For a small deformation f_t of f_0 we have that $f_t[V_\tau(t) \setminus \cup_j B_j(p_j)(t)]$ stays near $f[V_\tau \setminus \cup_j B_j(p_j)]$. Hence for t sufficiently small the components of expression 6.2 do not vanish both inside as well as outside of the neighborhood $\cup_j B_j(p_j)(t)$.

This implies that the components of f_t do not vanish along each generic fiber that extends locally as a singular curve of the foliation $\mathcal{F}_t$. This is possible only if f_t is constant along these curves. In fact, $f_t(V_\tau(t))$ is either a curve or a point. If it is a curve then it cuts all lines of $\mathbb{P}^2$ and therefore the components should be zero somewhere. Hence $f_t(V_\tau(t))$ is constant and we conclude that $V_\tau(t)$ is a fiber.

Observe also that when we make a blow-up with weights $(\gamma, \gamma, 1)$ at the points of $I(f_t)$ we solve completely the indeterminacy points of the mappings f_t for each t. $\square$

6.1.2. *Part 2.* Let us now define a family of foliations $(\mathcal{G}_t)_{t \in D_\epsilon}, \mathcal{G}_t \in \mathcal{A}$ (see Section 4) such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_\epsilon$. Firstly we consider the case $n = 3$. Instead of utilize the foliation $\mathcal{F}$ obtained as the foliation $f^*\mathcal{G}$, the idea that we will utilize in this part of the proof is to consider $\mathcal{F}$ on $\mathbb{P}^n$ defined as the foliation pull-back foliation from $\mathbb{P}^n$ to $\mathbb{P}^2_{[\gamma,\gamma,1]}$.

$$\begin{array}{rcl} \overline{f} : \mathbb{P}^n & \rightarrow & \mathbb{P}_{[\gamma,\gamma,1]} \\ \overline{f}^* \eta & \rightarrow & \eta. \end{array}$$

once they define the same foliation. Let $M_{[\gamma,\gamma,1]}(t)$ be the family of "complex algebraic threefolds" obtained from $\mathbb{P}^3$ by blowing-up with weights $(\gamma, \gamma, 1)$ at the $\frac{\nu^3}{\gamma}$ points $p_1(t), ..., p_j(t), ..., p_{\frac{\nu^3}{\gamma}}(t)$ corresponding to $I(t)$ of $\mathcal{F}_t$; and denote by

$$\pi_w(t) : M_{[\gamma,\gamma,1]}(t) \to \mathbb{P}^3$$

the blowing-up map. The exceptional divisor of $\pi_w(t)$ consists of $\frac{\nu^3}{\gamma}$ orbifolds $E_j(t) = \pi_w(t)^{-1}(p_j(t))$, $1 \leq j \leq \frac{\nu^3}{\gamma}$, which are weighted projective planes of the type $\mathbb{P}^2_{[\gamma,\gamma,1]}$. More precisely, if we blow-up $\mathcal{F}_t$ at the point $p_j(t)$, then the restriction of the strict transform $\pi_w^*\mathcal{F}_t$ to the exceptional divisor $E_j(t) = \mathbb{P}^2_{[\gamma,\gamma,1]}$ is the same quasi-homogeneous 1-form that defines $\mathcal{F}_t$ at the point $p_j(t)$. Using the map

$$\begin{array}{rcl} f_w : \mathbb{P}^2_{[\gamma,\gamma,1]} & \rightarrow & \mathbb{P}^2 \\ (x_0 : x_1 : x_2) & \rightarrow & (x_0 : x_1 : x_2^\gamma) \end{array}$$

it follows that we can push-forward the foliation to $\mathbb{P}^2$. Let us denote by $\mathbb{F}ol_2'[d', 2, (\gamma, \gamma, 1)]$ the set of $\{\hat{\mathcal{G}}\}$ saturated foliations of degree $d' = \gamma(d+1) + 1$ on $\mathbb{P}^2_{[\gamma,\gamma,1]}$ with one invariant line in general position and $Il_1(d, 2)$ the subsets of saturated foliations with an invariant line in $\mathbb{P}^2$ respectively. The mapping $f_w : \mathbb{P}^2_{[\gamma,\gamma,1]} \to \mathbb{P}^2$ induces a natural isomorphism $(f_w)_* : Il_1(d, 2) \to \mathbb{F}ol_2'[d', 2, (\gamma, \gamma, 1)]$. With this process in mind we produce a family of holomorphic foliations in $\mathcal{A} \subset Il_1(d, 2)$. This family is the "holomorphic path" of candidates to be a deformation of $\mathcal{G}_0$. In fact, since $(\mathcal{A}' = f_w)_*(\mathcal{A})$ is an open set inside $\mathbb{F}ol_2'[d', 2, (\gamma, \gamma, 1)]$ we can suppose that this family is inside $\mathcal{A}$. Hence using the mapping f_{w*} we can transport holomorphic from $\mathcal{A}$ to $\mathcal{A}'$ and vice-versa.

We fix the exceptional divisor $E_1(t)$ to work with and we denote by $\hat{\mathcal{G}}_t \in \mathcal{A}'$ the restriction of $\pi_w^*\mathcal{F}_t$ to $E_1(t)$. As we have seen, this process produces foliations in $\mathcal{A}'$ up to a linear automorphism of $\mathbb{P}^2_{[\gamma,\gamma,1]}$. Consider the family of mappings

$\overline{f}_t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2_{[\gamma,\gamma,1]}$, $t \in D_{\epsilon'}$ defined in Proposition 6.1. We will consider the family $(\overline{f}_t)_{t\in D_\epsilon}$ as a family of rational maps $\overline{f}_t : \mathbb{P}^3 \dashrightarrow E_1(t)$; we decrease ϵ if necessary. Note that the map

$$\overline{f}_t \circ \pi_w(t) : M_{[\gamma,\gamma,1]}(t) \backslash \cup_j E_j(t) \to E_1(t) \simeq P^2_{[\gamma,\gamma,1]}$$

extends holomorphically, that is, as an orbifold mapping, to

$$\hat{f}_t : M_{[\gamma,\gamma,1]}(t) \to E_1(t) \simeq \mathbb{P}^2_{[\gamma,\gamma,1]}.$$

This is due to the fact that each orbit of the vector field S_t determines an equivalence class in $\mathbb{P}^2_{[\gamma,\gamma,1]}$ and is a fiber of the map

$$(x_0(t), x_1(t), x_2(t)) \to (x_0(t), x_1(t), x_2^\gamma(t)).$$

The mapping $\overline{f}_t$ can be interpreted as follows. Each fiber of $\overline{f}_t$ meets $p_j(t)$ once, which implies that each fiber of $\hat{f}_t$ cuts $E_1(t)$ once outside of the singular line in $[M_{[\gamma,\gamma,1]}(t) \cap E_1(t)]$. Since $M_{[\gamma,\gamma,1]}(t) \backslash \cup_j E_j(t)$ is biholomorphic to $\mathbb{P}^3 \backslash I(t)$, after identifying $E_1(t)$ with $\mathbb{P}^2_{[\gamma,\gamma,1]}$, we can imagine that if $q \in M_{[\gamma,\gamma,1]}(t) \backslash \cup_j E_j(t)$ then $\hat{f}_t(q)$ is the intersection point of the fiber $\hat{f}_t^{-1}(\hat{f}_t(q))$ with $E_1(t)$. We obtain a mapping

$$\hat{f}_t : M_{[\gamma,\gamma,1]}(t) \to \mathbb{P}_{[\gamma,\gamma,1]}.$$

It can be extended over the singular set of $M_{[\gamma,\gamma,1]}(t)$ using Riemann's Extension Theorem. This is due to the fact that the orbifold $M_{[\gamma,\gamma,1]}(t)$ has singular set of codimension 2 and these singularities are of the quotient type; therefore it is a normal complex space. We shall also denote this extension by $\hat{f}_t$ to simplify the notation. We remark that the blowing-up with weights $(\gamma, \gamma, 1)$ can completely solve the indeterminacy set of $\overline{f}_t$ or f_t for each t as the reader can check. With all these ingredients we can define the foliation $\tilde{\mathcal{F}}_t = \overline{f}_t^*(\hat{\mathcal{G}}_t) = f_t^*(\mathcal{G}_t) \in PB(\Gamma - 1, \nu, 1, 1, \gamma)$. This foliation is a deformation of $\mathcal{F}_0$. Based on the previous discussion let us denote $\mathcal{F}_1(t) = \pi_w(t)^*(\mathcal{F}_t)$ and $\hat{\mathcal{F}}_1(t) = \pi_w(t)^*(\tilde{\mathcal{F}}_t)$.

Lemma 6.5. *If $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$ are the foliations defined previously, we have that*

$$\mathcal{F}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\gamma,\gamma,1]}} = \hat{\mathcal{G}}_t = \hat{\mathcal{F}}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\gamma,\gamma,1]}}$$

where $\hat{\mathcal{G}}_t$ is the foliation induced on $E_1(t) \simeq \mathbb{P}^2_{[\gamma,\gamma,1]}$ by the quasi-homogeneous 1-form $\eta_{p_1(t)}$.

Proof. In a neighborhood of $p_1(t) \in I(t)$, $\mathcal{F}_t$ is represented by the quasi-homogeneous 1-form $\eta_{p_1(t)}$. This 1-form satisfies $i_{S_t}\eta_{p_1(t)} = 0$ and therefore naturally defines a foliation on the weighted projective space $E_1(t) \simeq \mathbb{P}^2_{[\gamma,\gamma,1]}$. This proves the first equality. The second equality follows from the geometrical interpretation of the mapping $\hat{f}_t : M_{[\gamma,\gamma,1]}(t) \to \mathbb{P}^2_{[\gamma,\gamma,1]}$, since $\hat{\mathcal{F}}_1(t) = \hat{f}_t^*(\hat{\mathcal{G}}_t)$. □

Now we choose an afine chart of the space $\mathbb{P}^2_{[\gamma,\gamma,1]}$. This afine chart is biholomorphic to $\mathbb{C}^2$. In this affine chart for each t the foliation $(\hat{\mathcal{G}}_t)$ has d^2 singular points.

Let $\tau_1(t)$ be a singularity of $\hat{\mathcal{G}}_t$ outside of the line at infinity . Since the map $t \to \tau_1(t) \in \mathbb{P}^2_{[\gamma,\gamma,1}$ is holomorphic, there exists a holomorphic family of automorphisms of $\mathbb{P}^2_{[\gamma,\gamma,1}$, $t \to H(t)$ such that $\tau_1(t) = [0:0:1] \in E_1(t) \simeq \mathbb{P}^2_{[\gamma,\gamma,1}$ is kept fixed. Observe that such a singularity has non algebraic separatrices at this point. Fix

a local analytic coordinate system (x_t, y_t) at $\tau_1(t)$ such that the local separatrices are $(x_t = 0)$ and $(y_t = 0)$, respectively. Here we are considering the affine chart of $\mathbb{P}^2_{[\gamma,\gamma,1}$ which is biholomorphic to $\mathbb{C}^2$. This is useful because the foliations $\mathcal{G}_t$ and $\hat{\mathcal{G}}_t$ in this local coordinates are at least bihomolomorphic equivalents. Observe that the local smooth hypersurfaces along $\hat{V}_{\tau_1(t)} = \hat{f}_t^{-1}(\tau_1(t))$ defined by $\hat{X}_t := (x_t \circ \hat{f}_t = 0)$ and $\hat{Y}_t := (y_t \circ \hat{f}_t = 0)$ are invariant for $\hat{\mathcal{F}}_1(t)$. Furthermore, they meet transversely along $\hat{V}_{\tau_1(t)}$. On the other hand, $\hat{V}_{\tau_1(t)}$ is also contained in the Kupka set of $\mathcal{F}_1(t)$. Therefore there are two local smooth hypersurfaces $X_t := (x_t \circ \hat{f}_t = 0)$ and $Y_t := (y_t \circ \hat{f}_t = 0)$ invariant for $\mathcal{F}_1(t)$ such that:

(1) X_t and Y_t meet transversely along $\hat{V}_{\tau_1(t)}$.
(2) $X_t \cap \pi_w(t)^{-1}(p_1(t)) = (x_t = 0) = \hat{X}_t \cap \pi_w(t)^{-1}(p_1(t))$ and $Y_t \cap \pi_w(t)^{-1}(p_1(t)) = (y_t = 0) = \hat{Y}_t \cap \pi_w(t)^{-1}(p_1(t))$ (because $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$) coincide on $E_1(t) \simeq \mathbb{P}^2$).
(3) X_t and Y_t are deformations of $X_0 = \hat{X}_0$ and $Y_0 = \hat{Y}_0$, respectively.

Lemma 6.6. $X_t = \hat{X}_t$ *for small* t.

Proof. Let us consider the projection $\hat{f}_t : M_{[\gamma,\gamma,1]}(t) \to \mathbb{P}^2_{[\gamma,\gamma,1]}$ on a neighborhood of the regular fibre $\hat{V}_{\tau_1(t)}$, and fix local coordinates x_t, y_t on $\mathbb{P}^2_{[\gamma,\gamma,1}$ such that $X_t := (x_t \circ \hat{f}_t = 0)$. For small ϵ, let $H_\epsilon = (y_t \circ \hat{f}_t = \epsilon)$. Thus $\hat{\Sigma}_\epsilon = \hat{X}_t \cap H_\epsilon$ are (vertical) compact curves, deformations of $\hat{\Sigma}_0 = \hat{V}_{\tau_1(t)}$. Set $\Sigma_\epsilon = X_t \cap \hat{H}_\epsilon$. The $\Sigma'_\epsilon s$, as the $\hat{\Sigma}'_\epsilon s$, are compact curves (for t and ϵ small), since X_t and $\hat{X}_t$ are both deformations of the same X_0. Thus for small t, X_t is close to $\hat{X}_t$. It follows that $\hat{f}_t(\Sigma_\epsilon)$ is an analytic curve contained in a small neighborhood of $\tau_1(t)$, for small ϵ. By the maximum principle, we must have that $\hat{f}_t(\Sigma_\epsilon)$ is a point, so that $\hat{f}_t(X_t) = \hat{f}_t(\cup_\epsilon \Sigma_\epsilon)$ is a curve C, that is, $X_t = \hat{f}_t^{-1}(C)$. But X_t and $\hat{X}_t$ intersect the exceptional divisor $E_1(t) = \mathbb{P}^2_{[\gamma,\gamma,1}$ along the separatrix $(x_t = 0)$ of $\mathcal{G}_t$ through $\tau_1(t)$. This implies that $X_t = \hat{f}_t^{-1}(C) = \hat{f}_t^{-1}(x_t = 0) = \hat{X}_t$. □

We have proved that the foliations $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ have a common local leaf: the leaf that contains $\pi_w(t)\left(X_t \backslash \hat{V}_{\tau_1(t)}\right)$ which is not algebraic. Let $D(t) := Tang(\mathcal{F}(t), \hat{\mathcal{F}}(t))$ be the set of tangencies between $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$. This set can be defined by $D(t) = \{Z \in \mathbb{C}^4; \Omega(t) \wedge \hat{\Omega}(t) = 0\}$, where $\Omega(t)$ and $\hat{\Omega}(t)$ define $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$, respectively. Hence it is an algebraic set. Since this set contains an immersed non-algebraic surface X_t, we necessarily have that $D(t) = \mathbb{P}^3$. This proves Theorem B in the case $n = 3$.

Suppose now that $n \geq 4$. The previous argument implies that if Υ is a generic $3-$plane in $\mathbb{P}^n$, we have $\mathcal{F}(t)_{|\Upsilon} = \hat{\mathcal{F}}(t)_{|\Upsilon}$. In fact, such planes cut transversely every strata of the singular set, and $I(t)$ consists of $\frac{\nu^3}{\gamma}$ points. This implies that f_t is generic for $|t|$ sufficiently small. We can then repeat the previous argument, finishing the proof of Theorem A.

Recall from Definition 2.2 the concept of a generic map. Let $f \in BRM\,(n, \nu, 1, 1, \gamma)$, $I(f)$ its indeterminacy locus and $\mathcal{F}$ a foliation on $\mathbb{P}^n$, $n \geq 3$. Consider the following properties:

$\mathcal{P}_1$: If n=3, at any point $p_j \in I(f)$ $\mathcal{F}$ has the following local structure: there exists an analytic coordinate system (U^{p_j}, Z^{p_j}) around p_j such that $Z^{p_j}(p_j) = 0 \in (\mathbb{C}^3, 0)$ and $\mathcal{F}|_{(U^{p_j}, Z^{p_j})}$ can be represented by a quasi-homogenous 1-form η_{p_j} (as described in the Lemma 5.7) such that

(a) $Sing(d\eta_{p_j}) = 0$,
(b) 0 is a quasi-homogeneous singularity of the type $[\gamma : \gamma : 1]$.

If $n \geq 4$, $\mathcal{F}$ has a local structure product: the situation for n=3 "times" a regular foliation in $\mathbb{C}^{n-3}$.

$\mathcal{P}_2$: There exists a fibre $f^{-1}(q) = V(q)$ such that $V(q) = f^{-1}(q) \backslash I(f)$ is contained in the Kupka-Set of $\mathcal{F}$ and $V(q)$ is not contained in $(F_2 = 0)$.

$\mathcal{P}_3$: $V(q)$ has transversal type X, where X is a germ of vector field on $(\mathbb{C}^2, 0)$ with a non algebraic separatrix and such that $0 \in \mathbb{C}^2$ is a non-degenerate singularity with eigenvalues λ_1 and λ_2, $\frac{\lambda_2}{\lambda_1} \notin \mathbb{R}$.

Lemma 6.6 allows us to prove the following results:

Theorem B. *In the conditions above, if properties $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ hold then $\mathcal{F}$ is a pull back foliation, $\mathcal{F} = f^*(\mathcal{G})$, where $\mathcal{G}$ is of degree $d \geq 2$ on $\mathbb{P}^2$ with one invariant line.*

Let us denote by $\mathbb{F}ol_2'[d', 2, (\gamma, \gamma, 1)]$ the set of $\{\hat{\mathcal{G}}\}$ saturated foliations of degree d' on $\mathbb{P}^2_{[\gamma,\gamma,1]}$ with one invariant line. According to this notation the previous Thereom can be re-written as:

Theorem C. *In the conditions above, if properties $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ hold then $\mathcal{F}$ is a pull back foliation, $\mathcal{F} = \overline{f}^*(\hat{\mathcal{G}})$, where $\hat{\mathcal{G}} \in \mathbb{F}ol_2'[d', 2, (\gamma, \gamma, 1)]$*

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Current address: IRMAR Université de Rennes 1, Campus de Beaulieu 35042 Rennes Cedex France

E-mail address: `wancossil@gmail.com`